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From (12) we have

$$R = g^{pr}R_{pr} = n\varphi,$$

and (12) becomes

$$R_{pr} = \frac{1}{n}\, g_{pr}R. \tag{13}$$

The original Einstein equations (1914) for space free from matter are those for which $\varphi = 0$ in (12); in 1917 (*Sitz. Pr. Ak. Wiss.*, Feb. 8), those for which $\varphi =$ const.; and in 1919 (*Ibid.*, Apr. 10), the general case (13) of a homogeneous space from the above point of view.

From (8), (9) and (12) it follows that for any three mutually orthogonal directions in any 3-space, which is homogeneous,

$$r_{12} + r_{13} = r_{21} + r_{23} = r_{31} + r_{32} = -\varphi,$$

whence

$$r_{12} = r_{13} = r_{23} = -\frac{\varphi}{2}.$$

Thus the Riemann curvature at each point is the same for all directions, and by the theorem of Schur (*Math. Ann.*, **27**, 563) is constant. Consequently the first type of Einstein space is a generalization of euclidean 3-space, and his other two spaces of 3-space of constant curvature.

NOTE ON THE DEFINITION OF A LINEAL FUNCTIONAL

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A linear functional is usually defined as one that is distributive and continuous, but the term continuous functional has been used in at least two ways which are not equivalent. F. Riesz, Fréchet, G. C. Evans and others have defined a continuous functional as one which satisfies the equation

$$\lim_{n \to \infty} L(u_n(x)) = L(u(x)), \tag{1}$$

when the sequence $u_1(x)$, $u_2(x)$, ... approaches $u(x)$ uniformly, while Levy[1] and W. L. Hart[2] have simply assumed that the sequence of u_n's converge in the mean. In Levy's paper the functional also depends on a parameter which has the same range as x, and $L(u_n(x),y)$ is only required to converge to $L(u(x),y)$ in the mean, but such a parameter will not be introduced here.

In what follows a distributive functional will be called linear when equation (1) is satisfied for uniformly convergent sequences of u's, and linear-*m* when they are only required to converge in the mean. If then L is to be linear it is necessary and sufficient that the equation

$$L(au_1 + bu_2) = aL(u_1) + bL(u_2) \tag{2}$$

shall be satisfied, and that there be an M independent of u which satisfies the inequality

$$|L(u)| \leqq M . max|u|, \tag{3}$$

while if it is to be linear-m this inequality must be replaced by

$$|L(u)| \leqq M \sqrt{\int_a^b u^2(x) dx}. \tag{4}$$

F. Riesz has proved that every linear functional can be put into the Stieltjes form

$$L(u) = \int_a^b u(x) d\alpha(x), \tag{5}$$

where $\alpha(x)$ is independent of u and has finite variation.[3] Fréchet has reduced this to the form

$$L(u) = \sum_{n=1}^{\infty} A_n u(x_n) + \int_a^b u(x)\beta(x)dx + \int_a^b u(x)d\lambda(x), \tag{6}$$

where the A_n's are constants, the x_n's are the points where $\alpha(x)$ is discontinuous, $\beta(x)$ is the derivative of $\alpha(x)$ where the derivative exists, and $\lambda(x)$ is continuous, and has its derivative equal to zero excepting on a set of measure zero.[4] The principal object of the present note is to prove that when L is linear-m equation (6) becomes simply

$$L(u) = \int_a^b u(x)\beta(x)dx. \tag{7}$$

The class $[u(x)]$ will include all bounded, real-valued functions defined on (a, b), which are integrable with respect to a function of finite variation by Young's method of monotone sequences, beginning with the set of continuous functions.[5]

In the proof of equation (5) it was only assumed that $L(u)$ is defined for continuous u's, and to make the theorem apply to the larger class, L must always be defined in such a way that whenever a monotone sequence $\{u_n(x)\}$ approaches a discontinuous $u(x)$, $L(u_n)$ will approach $L(u)$. When this is done equation (6) will also be satisfied for the larger domain $[u(x)]$, since the limiting processes extending the definition of the Stieltjes integral (5) to the domain $[u(x)]$ will make each term of equation (6) approach the proper value. On the other hand if such a monotone sequence approaches $u(x)$, the equation

$$\operatorname*{limit}_{n \to \infty} \int_a^b (u_n - u)^2 dx = 0$$

must be satisfied.[6] Then if L is linear-m inequality (4) implies that $L(u_n - u)$ will approach zero, and $L(u_n)$ must approach $L(u)$.

Since a uniformly convergent sequence must converge in the mean, a linear-m functional must also be linear.

If $L(u)$ is linear-m, and $u(x)$ is defined as identically zero excepting at x_1 where it is equal to unity, equation (6) becomes $L(u) = A_1$, and inequality (4) implies that $A_1 = 0$. Similarly $A_2 = A_3 = \ldots\ldots = 0$. In other words $\alpha(x)$ must be continuous.

Since $\alpha(x)$ also has finite variation, it can be put into the form $\alpha(x) = \alpha_1(x) - \alpha_2(x)$, where α_1 and α_2 are continuous, have finite variation, and are monotone increasing. The derivatives of α_1 and α_2 will be called β_1 and β_2. A set E of measure zero will contain all points where either β fails to exist. Since no derivative number of a monotone increasing function can be negative, the functions $\lambda_i(x)$, defined by the equations

$$\lambda_i(x) = \alpha_i(x) - \int_a^x \beta_i(x)dx, \qquad (i = 1, 2) \tag{8}$$

must be monotone increasing,[7] and their derivatives will vanish excepting on E. If a $k_1 > 0$ is now given, there must be a sequence of non-overlapping open intervals, $B_1 = \sum_{n=1}^{\infty} I_n$ containing each point of E as an interior point of some I_n, such that the measure of B_1 is less than k_1. Then the variation of λ_i on B_1 is equal to its variation on (a, b),[8] which is $\lambda_i(b) - \lambda_i(a)$. The lower semicontinuous functions $u_1(x)$, $u_2(x), \ldots\ldots$ will now be defined by the equations

$u_n(x) = 1$ on the interior of the intervals $I_1, I_2, \ldots\ldots I_n$,
$u_n(x) = 0$ for all other values of x,

and consequently

$$\int_a^b u_n(x)d\lambda_i(x) = \text{Variation of } \lambda_i \text{ on } \sum_{m=1}^{n} I_m, \quad (i = 1, 2;\ n = 1, 2, \ldots).$$

The limit $v_1(x)$ of this monotone sequence of u's, which vanishes excepting on B_1, must satisfy the equations

$$\int_a^b v_i(x)d\lambda_i(x) = \lambda_i(b) - \lambda_i(a). \qquad (i = 1, 2)$$

If a sequence $k_1 > k_2 > \ldots\ldots \longrightarrow 0$ is chosen, beginning with the above k_1, the sequences of intervals $B_2, B_3, \ldots\ldots$ can be chosen in such a way that each contains E, every point of each is a point of the preceding, and the measure of B_j is less that k_j. Then the sequence $v_2(x), v_3(x), \ldots\ldots$, determined as $v_1(x)$ was, will be monotone decreasing, and its limit $v(x)$ will vanish excepting on a set of measure zero, and satisfy the equations

$$\int_a^b v(x)d\lambda_i(x) = \lambda_i(b) - \lambda_i(a). \qquad (i = 1, 2) \tag{9}$$

Since $v(x)$ vanishes excepting on a set of measure zero, the equations

$$\int_a^b v(x)\beta_i(x)dx = 0, \qquad (i = 1, 2)$$

must be satisfied, and inequality (4) implies that $L(v) = 0$. But since $L(u)$ must satisfy equation (6) which is now equivalent to

$$L(u) = \int_a^b u(x)(\beta_1(x) - \beta_2(x))dx + \int_a^b u(x)d(\lambda_1(x) - \lambda_2(x)), \qquad (10)$$

it is necessary that

$$\int_a^b v(x)d(\lambda_1(x) - \lambda_2(x)) = 0.$$

Then equation (9) implies that $\lambda_1(b) - \lambda_2(b) = \lambda_1(a) - \lambda_2(a)$.

If in the above proof the functions $u_n(x)$, $v_j(x)$, and $v(x)$ had been defined as they were on the interval (a, y), where y is an arbitrary point on (a, b), and as identically zero on the rest of (a, b), the same argument would prove that $\lambda_1(y) - \lambda_2(y) = \lambda_1(a) - \lambda_2(a)$. That is, $\lambda_1 - \lambda_2$ is a constant, and equation (10) is equivalent to equation (7), with $\beta = \beta_1 - \beta_2$. This completes the proof of the theorem.

If L also depends on a parameter, y, and if instead of assuming that equation (1) is satisfied for every value of y, it is assumed that $L(u_n(x),y)$ approaches $L(u(x),y)$ in the mean, the theorem does not apply. For instance the functional

$$L(u(x),y) = u(y)$$

satisfies this condition, but cannot be represented by an integral analogous to (7).

[1] Levy, *Bul. Soc. Math. France,* **48,** 1920 (20).

[2] Hart, *New York, Bul. Amer. Math. Soc.,* **27,** 1921 (308).

[3] F. Riesz, *Ann. Sci. Ec. Norm., Paris* (Ser. 3), **31,** 1914 (9–14).

[4] Fréchet, *New York, Trans. Amer. Math. Soc.,* **15,** 1914 (152), or Evans, *Cambridge (Mass.), Colloquium Lectures* (60).

[5] Young, *Proc. London Math. Soc.,* **13,** 1914 (137), or Daniell, *Princeton, Ann. Math.,* **19,** 1917 (288–93).

[6] Bliss, *New York, Bul. Amer. Math. Soc.,* **24,** 1917 (24, corollary).

[7] *Ibid.* (38, Lemma 8).

[8] *Ibid.* (36–40, Theorems, 18 and 19).

SKIN TRANSPLANTATION IN FROG TADPOLES

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1. The experimental inhibition of vision by means of opaque grafts over the eyes of frog tadpoles (from 20 to 100 mm. in length) should contribute evidence as to any regulatory interaction between the graft and the eye, tending to restore the function of the eye. For this purpose 202 operations were performed upon two species of Rana, catesbeiana and clamatans, which had been collected from three widely separated sources.